International Trade and Currency Exchange

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Abstract

On the international scene, away from national legal rules, the use of different currencies is largely due to the operation of the "Invisible Hand". The paper develops a three-country model of the world economy. This links real trade patterns with currency exchange structures in a general equilibrium framework which includes transaction costs on foreign exchange markets. In the presence of strategic complementarities, there are multiple equilibrium structures of currency exchange for a given underlying real trade pattern. The existence conditions of these different equilibria are characterised, using the trade links between countries as the key parameters. Finally, repercussions on world output of the choice of a currency exchange structure are analysed.

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"So much of barbarism, however, still remains in the transactions of most civilized nations, that almost all independent countries choose to assert their nationality by having, to their own inconvenience and that of their neighbours, a peculiar currency of their own."
John Stuart Mill, 1848.

1. Introduction

The internationalisation of the pound began early in the 1800s and continued for more than a century. The industrial revolution transformed Britain into the world's richest economy and leading trading nation. The years after the First World War saw the decline of Britain as an international power, but sterling kept a lot of its functions. Only after the Second World War did the decline of sterling accelerate, with a sharp rise in the use of the dollar as an international currency.

Much more than a national currency, whose status is usually enforced by a set of legal restrictions, the use of currencies as international media of exchange is largely determined by the "Invisible Hand". Yet the process of internationalisation of currencies has not been satisfactorily explained so far, and the difficulty of modelling money and currency exchange is a recurrent theme in economic theory (see e.g. Hellwig [1992]).

The paper concerns a distinct aspect of the internationalisation of currencies: the role of a *vehicle* currency, i.e. a third currency against which each of two other currencies is consecutively exchanged in order to make the net exchange at lower cost than a direct exchange would entail. It seeks to explain the rise and fall of vehicle currencies through evolutions in the pattern of real trade between countries. An original feature of the model is that financial intermediaries operate on different bilateral exchange markets whose efficiency depends on their "thickness". There is a strategic complementary in the exchange process. This "thick market externality" permits retaining the main intuition of search models while making it possible to deal with prices and exchange rates in an analytically tractable manner.

Currency internationalisation has been discussed in various contexts - see for example Cohen [1971], McKinnon [1979], Krugman [1984], Alogoskoufis and Portes [1993] - but very rarely modelled. Matsuyama, Kiyotaki, Matsui [1993] and Zhou [1997] are notable exceptions. These two papers build on the random matching models of money as developed by Kiyotaki and Wright [1989] . They look at the internationalisation of a currency from the perspective of currency substitution. In reality, however, currency substitution is very often linked to high inflation, whereas in these models prices are rigid. They also postulate the indivisibility of money[1], which prevents them from determining prices and exchange rates endogenously. Moreover, currency exchange arises only in a mixed strategy equilibrium in Matsuyama, Kiyotaki, Matsui [1993]. Zhou [1997] gets currency exchange in pure strategies equilibria only at the cost of assuming random taste reversals between the domestic and foreign goods. Finally, being two-country models, they rule out the possibility of indirect exchange, which is at the root of the existence of vehicle currencies. Vehicle currencies have been studied by Krugman [1980] in a static, partial equilibrium environment and more recently, in the context of the foreign exchange market microstructure literature, by Hartmann [1998].

The paper links explicitly the emergence of vehicle currencies to the pattern of international trade. In section 2 we present the model and describe the structures of currency exchange that can arise from the different patterns of world trade. In section 3, we solve the model. An interesting result is the existence of multiple equilibria for a wide range of the parameters. We show how these structures can be related to the degree of trading dominance of the different countries. Section 4 uses the model to explain the rise and fall of international currencies and to account for inertia in their use, a phenomenon that has been very often described but never formalised. In particular, we discuss the case of sterling and the dollar. Section 5 evaluates the real costs linked to the different patterns of exchange. Finally section 6 discusses the modelling strategy, and section 7 concludes.

2. Model

2.1. Physical environment

The model incorporates classic ingredients of deterministic cash-in-advance models in an open economy framework[2] but adds an explicit transaction technology. It also builds on the work of Krugman [1980].

There are three countries fully specialized in production, three currencies, three perishable commodities and three one-period discount bonds denominated

[1] Shi [1997] has relaxed this assumption in a closed economy framework.
[2] See e.g. Lucas (1990), Grilli-Roubini (1993).

in the three currencies. There is a government in each country which prints money and issues bonds denominated in domestic currency. There is no trade in bonds, only trade in goods[3]. The production technology is a constant return to scale technology with labour as sole input. Agents are infinitely lived and can be either producers or financial intermediaries. There is no labour mobility between countries but full labour mobility between sectors within a country. Both producers and intermediaries maximize profit, have the same preferences and consume. An agent cannot consume her own production. There is a cash-in-advance constraint on the goods market. Goods have to be paid for in the currency of the seller[4]. Consumers cannot engage in foreign exchange operations themselves but have to request the services of a financial intermediary. They pay for the use of this transaction technology. Consumers deal exclusively with intermediaries of their own country. But financial intermediaries of a given country can operate on any bilateral foreign exchange market. The cost of currency exchange on a particular bilateral market is decreasing with the volume exchanged on that market ("thick market externality")[5]. Intermediaries can be thought of as operating on wholesale foreign exchange markets while interacting with consumers on retail markets. The transaction process draws resources out of production; thus the economy incurs real costs whose magnitudes depend on the "efficiency" of the transaction technology.

2.2. Time structure

The lives of the agents are divided into discrete time periods. Each time period is divided into two subperiods: the first one for currency exchange, the second one for goods purchases. At the beginning of a period, production takes place. Foreign exchange markets open. Consumers make consumption and savings decisions. They acquire via financial intermediaries the portfolio of monies necessary for their good purchase programmes. Foreign exchange markets close and good

[3] Bonds are introduced only to insure that monies are dominated as store of values. Any interest-bearing storage technology could have been used. It turned out that using government bonds was the easiest. Linking internationalisation to both trade in goods and in capital is beyond the scope of this paper. See a more detailed discussion of this issue in section 6.

[4] We will comment on that assumption in section 6.

[5] There is evidence pointing to economies of scale on foreign exchange markets. Several studies have found a negative correlation between bid-ask spreads and predetermined volumes exchanged on a given market (Bessembinder [1994], Hartmann [1998] use time series data; Black [1991], Hartmann [1998] use short panels). For a more detailed discussion of the evidence see Portes and Rey [1998].

markets open. Consumers pay producers and workers receive their labour income in domestic currency. Intermediaries are also paid in domestic currency by the consumers for the transaction services of this period[6]. Consumption takes place. And the next period begins.

2.3. Agents

In what follows the three countries are indexed by $\ell \in \{i, j, k\}$, t is a time subscript.

Government

The government ℓ prints money (the rate of growth of money ℓ is $\rho_{\ell,t} = \frac{M_{\ell,t+1} - M_{\ell,t}}{M_{\ell,t}}$). It issues one period-nominal bonds in the amount of $B_{\ell,t}$. The price of the bonds in period t is $q_{\ell,t} = \frac{1}{1+i_{\ell,t}}$ where $i_{\ell,t}$ is the nominal interest rate.

Consumers

Preferences

The intertemporal utility function for consumers of country ℓ is

$$\begin{aligned}
\Phi^{\ell} &= \sum_{t=0}^{\infty} \beta^t U^{\ell}\left(C_{i,t}^{\ell}, C_{j,t}^{\ell}, C_{k,t}^{\ell}\right) \text{ with} \\
U^{\ell}\left(C_{i,t}^{\ell}, C_{j,t}^{\ell}, C_{k,t}^{\ell}\right) &= \alpha_i^{\ell} \ln(C_{i,t}^{\ell}) + \alpha_j^{\ell} \ln(C_{j,t}^{\ell}) + \alpha_k^{\ell} \ln(C_{k,t}^{\ell}) \\
\text{and } \sum_j \alpha_j^{\ell} &= 1 \text{ for all } \ell
\end{aligned}$$

where $C_{i,t}^{\ell}$ is the consumption of good i by a representative consumer of country ℓ in period t[7]. The discount factor β is assumed to be the same in the three countries.

Cash-in-advance constraints

At the beginning of period t, the typical consumer of country ℓ faces the following cash-in-advance constraint on the market for good i[8]:

$$M_{i,t}^{\ell} \geq P_{i,t} C_{i,t}^{\ell}$$

[6] There is no cash-in-advance constraint for the payment of the transaction service, nor to buy bonds.

[7] See section 6 for a discussion of the form of the utility function.

[8] She faces similar constraints on the markets for goods j and k.

$M^{\ell}_{i,t}$ is the total amount of money of country i held by the consumers of country ℓ at time t and $P_{i,t}$ is the price of good i in currency i at date t.

Law of motion of wealth and budget constraint

We define $\tilde{e}^{ij}_t$ as the nominal spot exchange rate of currency i against currency j (number of units of currency i per unit of currency j). By convention $\tilde{e}^{ii}_t = 1$.

At the beginning of period t, the wealth Ψ^{ℓ}_t of the representative consumer of country ℓ is given by:

$$\Psi^{\ell}_t = \left[\sum_i \left(M^{\ell}_{i,t-1} - P_{i,t-1}C^{\ell}_{i,t-1}\right)\tilde{e}^{\ell i}_t\right] + P_{\ell,t-1}Y_{\ell,t-1} + B^{\ell}_{\ell,t-1}$$

where $P_{\ell,t-1}Y_{\ell,t-1}$ is the revenue from last period's sales. The income of the financial intermediation activity does not appear in this identity since the financial intermediation sector is modelled as an intermediate input sector.

The representative consumer divides her wealth between money holdings and bonds, and her budget constraint can be written as :

$$q_{\ell,t}B^{\ell}_{\ell,t} + \sum_i M^{\ell}_{i,t}e^{\ell i}_t \leq \Psi^{\ell}_t$$

where $q_{\ell,t}B^{\ell}_{\ell,t}$ is the amount of bonds of country ℓ purchased at date t (redeemed into domestic currency at date t+1).

We normalise each nominal variable by the corresponding money stock (for all i and j):

$$e^{ij}_t = \tilde{e}^{ij}_t M_{j,t}/M_{i,t};\, b^{i}_{j,t} = B^{i}_{j,t}/M_{j,t};\; m^{j}_{i,t} = M^{j}_{i,t}/M_{i,t};\; p_{i,t} = P_{i,t}/M_{i,t};\, \psi_{i,t} = \Psi_{i,t}/M_{i,t}$$

We can rewrite the optimisation problem of the consumer of country ℓ. Let z^{ℓ} be the value function for a consumer beginning the period with wealth ψ^{ℓ}_t.

$$\mathsf{z}^{\ell}(\psi^{\ell}_t) = \max_{\left\{C^{\ell}_{i,t}\right\}_{\mathrm{i}\in\{i,j,k\}}, b^{\ell}_{\ell,t}} \left[U(C^{\ell}_{i,t}, C^{\ell}_{j,t}, C^{\ell}_{k,t}) + \beta\,\mathsf{z}^{\ell}(\psi^{\ell}_{t+1})\right] \tag{2.1}$$

with

$$\psi^{\ell}_{t+1} = \left[\sum_i \left(m^{\ell}_{i,t} - p_{i,t}C^{\ell}_{i,t}\right)\frac{e^{\ell i}_{t+1}}{1+\rho_{i,t}}\right] + \frac{p_{\ell,t}Y_{\ell,t}}{1+\rho_{\ell,t}} + \frac{b^{\ell}_{\ell,t}}{(1+\rho_{\ell,t})} \tag{2.2}$$

subject to the budget constraint

$$q_{\ell,t} b_{\ell,t}^{\ell} + \sum_i m_{i,t}^{\ell} e_t^{i\ell} \leq \psi_t^{\ell} \tag{2.3}$$

the cash-in-advance constraints

$$p_{i,t} C_{i,t}^{\ell} \leq m_{i,t}^{\ell} \text{ for all i} \in \{i, j, k\} \tag{2.4}$$

and the non-negativity constraints

$$m_{j,t}^{i} \geq 0; C_{j,t}^{i} \geq 0 \text{ for all i, j} \in \{i, j, k\} \tag{2.5}$$

To rule out Ponzi schemes, we require:

$$\underset{t \to \infty}{lim} \left(\prod_j^t q_{\ell,j} \right) b_{\ell,t}^{\ell} = 0 \tag{2.6}$$

2.4. Technology

Production technology
The production technology has *constant returns to scale* in labour and is *country* specific. Total output $Y_{\ell,t}$ is given by:

$$Y_{\ell,t} = l_{\ell,t}^{P} \text{ for } \ell \in \{i, j, k\} \tag{2.7}$$

where $l_{\ell,t}^{P}$ is the number of agents of country ℓ involved in production in period t. The profit of producers $\pi_{\ell,t}^{P}$ is given by:

$$\pi_{\ell,t}^{P} = p_{\ell} Y_{\ell,t} - w_{\ell,t}^{P} l_{\ell,t}^{P} \tag{2.8}$$

where $w_{\ell,t}^{P}$ is the normalised wage in the production sector.

Transaction technology
The transaction technology *has increasing returns to scale* and is *market* specific. We assume that the transaction service required by country ℓ on foreign exchange market ij at date t to exchange currency i against currency j is proportional to a normalized measure $v_{ij,t}(\ell)$ of the volume of currency i exchanged against currency j[9]. We denote by $l_{ij,t}^{FI}(\ell)$ the number of intermediaries of country ℓ operating on

[9] For simplicity, we will take the coefficient of proportionality to be one.

market ij at date t and by $L_{ij,t}$ the total number of intermediaries who operate on market ij at date t[10]. The production of a transaction service $s_{ij,t}(\ell)$ for country ℓ takes the following form:

$$s_{ij,t}(\ell) = L_{ij,t}^{\alpha} l_{ij,t}^{FI}(\ell) \text{ for ij} \in \{ij, jk, ik\} \tag{2.9}$$

where α is a constant strictly greater than zero. The transaction technology is perfectly competitive from an individual financial intermediary's point of view but exhibits increasing returns in the aggregate. The greater is α, the greater is the thick market externality[11].

The profit of financial intermediaries $\pi_{\ell,t}^{FI}$ is given by:

$$\pi_{\ell,t}^{FI} = \sum_{ij} \left[T_{ij,t}^{\ell} s_{ij,t}(\ell) - w_{\ell,t}^{FI} l_{ij,t}^{FI}(\ell) \right] \tag{2.10}$$

where $T_{ij,t}^{\ell}$ is the unit price in country ℓ of a transaction service performed on bilateral market ij and $w_{\ell,t}^{FI}$ is the normalised wage in the financial intermediation sector.

Market clearing conditions

Goods markets:

$$\sum_{\ell} C_i^{\ell} = Y_i \text{ for i} \in \{i, j, k\} \tag{2.11}$$

Money markets:

$$\sum_{\ell} p_i C_i^{\ell} = 1 \text{ for i} \in \{i, j, k\} \tag{2.12}$$

Bond markets

$$b_{\ell}^{\ell} = b_{\ell} \text{ for all } \ell \tag{2.13}$$

Transaction services

$$v_{ij,t}(\ell) = s_{ij,t}(\ell) \text{ for all } \ell \text{ and all i,j} \tag{2.14}$$

Labour markets

$$\sum_{ij} l_{ij,t}^{FI}(\ell) + l_{\ell,t}^{P} = \Lambda_{\ell,t} \text{ for all } \ell \tag{2.15}$$

[10] Therefore $L_{ij,t} = \sum_{\ell} l_{ij,t}^{FI}(\ell)$

[11] This type of "Marshallian externality" has been extensively used in the endogenous growth literature.

where $\Lambda_{\ell,t}$ is the total labour endowment of country ℓ.

2.5. Different structures of currency exchange.

Utility and profit maximisations and the above market-clearing conditions do not suffice to characterise the equilibria. This is because for given money demands, there are several ways of exchanging currencies to clear the three bilateral foreign exchange markets. In figure 1, we represent bilateral trade flows between three countries I, J and K[12]. The arrow $\mathrm{I}\underset{\mathrm{F}}{\rightarrow}\mathrm{J}$ means that country I wants to exchange F units of currency i against the equivalent amount of currency j (in the case shown in figure 1, the corresponding flows from J to K and K to I are G and H respectively). Since we are in a three country world, current account equilibrium does not imply that *bilateral* trade flows are balanced[13]. But the "imbalance" (called D on figure 1) is the same on the three markets.

Definitions:

We call *partial indirect exchange with currency i as a vehicle currency*[14] (see figure 2) a structure where some exchange takes place directly between the currencies of J and K (G-D exactly) and some exchange takes place indirectly through the currency of I (D units of currency are exchanged indirectly).

We call *total indirect exchange with currency i as a vehicle currency* (see figure 3) a structure where all exchange between the currencies of countries J and K takes place indirectly through the currency of country I. Such an exchange structure leads to the complete disappearance of one of the three foreign exchange markets. In these two cases, the currency used to make indirect exchanges (here the currency of country I) is the vehicle currency. Partial and total indirect exchange structures with the currencies of countries J and K as vehicles are defined in an entirely analogous way.

Our equilibrium will have to satisfy rational expectations. When maximising profits, financial intermediaries take as given a particular structure of exchange (they take as given the $\{L_{ij}\}_{ij\in\{ij,ik,kj\}}$ [15]). Once the transaction costs are derived

[12] We associate all the variables with subscript i to country I, those with subscript j to country J and those with subscript k to country K.

[13] In other words, we may not have a "double coincidence of wants" on the foreign exchange markets.

[14] This terminology was first introduced by Krugman [1980].

[15] Number of financial intermediaries operating on each bilateral foreign exchange market ij.

in equilibrium, we will have to check that they are consistent with the structure of exchange that has been assumed.

2.6. Definition of equilibrium

We solve for stationary equilibria where the nominal variables grow at a constant rate in each of the three countries.
Let us denote by Θ the set $\{i, j, k\}$ and Φ the set $\{ij, ik, kj\}$.
A stationary equilibrium consists of goods and bond prices $\{p_\ell\}_{\ell\in\Theta}$, $\{q_\ell\}_{\ell\in\Theta}$, exchange rates $\left\{e^{i\ell}\right\}_{\ell\in\Theta}$, wages $\left\{w_\ell^P, w_\ell^{FI}\right\}_{\ell\in\Theta}$, transaction costs $\left\{T_{ij}^\ell\right\}_{\ell\in\Theta, ij\in\Phi}$, bond and money holdings $\left\{b_\ell^\ell\right\}_{\ell\in\Theta}$, $\left\{m_i^\ell\right\}_{i,\ell\in\Theta}$,consumption levels $\left\{C_i^\ell\right\}_{i,\ell\in\Theta}$,labour allocations $\left\{l_{ij}^{FI}(\ell), l_\ell^P\right\}_{\ell\in\Theta, ij\in\Phi}$, and value functions $\left\{\mathsf{z}^\ell\right\}_{\ell\in\Theta}$ such that:

(i) each z^ℓ satisfies (2.1) subject to the law of motion of wealth (2.2), the budget constraint (2.3), the cash-in-advance constraints (2.4), the non-negativity constraints (2.5), the no-Ponzi condition (2.6);

(ii) producers maximise profit (2.8) taking p_ℓ, w_ℓ^P as given, subject to the production technology (2.7);

(iii) financial intermediaries maximise profit (2.10) taking $\left\{T_{ij}^\ell\right\}_{ij\in\Phi}$, $\{L_{ij}\}_{ij\in\Phi}$, w_ℓ^{FI} as given subject to the transaction technology (2.9);

(iv) wages are equalised between the production and the financial intermediation sector: $w_\ell^P = w_\ell^{FI}$[16];

(v) goods, money, bond, transaction services, labour markets clear (2.11), (2.12), (2.13), (2.14), (2.15);

(vi) bilateral foreign exchange markets clear and equilibrium transaction costs are consistent with the assumed structure of exchange[17].

3. Solving the model

The main feature of the model is a non-trivial two-way relation between the monetary variables and the real variables. Trade in goods determines the money demands, and in turn currency exchange affects the division of labour between production and financial intermediation and therefore the relative prices.

[16] We assume full labour mobility within each country.
[17] See discussion of section 2.5.

3.1. First Order and Envelope Conditions

Assuming that monetary policy is such that cash-in-advance constraints are binding each period[18], the necessary conditions for the consumer of country ℓ are:

$$U_i = \lambda p_i; U_j = \lambda e^{ij} p_j; U_k = \lambda e^{ik} p_k; \mathsf{Z}_{\psi} = \lambda; \frac{\beta \mathsf{Z}_{\psi}}{q_\ell(1+\rho_\ell)} = \lambda \tag{3.1}$$

where λ is the multiplier associated with the budget constraint[19].
Hence we have the usual relation $q_\ell(1+\rho_\ell) = \beta$.
It is easy to show that the cash-in-advance constraints are binding if and only if monetary growth is such that $\rho_\ell > \beta - 1$ and the real interest rate is equal to the rate of time preference.

3.2. Price levels and exchange rates

From the money and the goods markets equilibrium, we have:

$$p_\ell = \frac{1}{Y_\ell} \tag{3.2}$$

Normalised price levels depend inversely on the level of output. Since production is affected by the amount of labour used in transactions, prices depend on the patterns of currency exchange: the more efficient the transaction technology, the lower the prices.
From the first order conditions (3.1) and the expression for the price levels (3.2), we can deduce the steady-state exchange rates[20]:

$$e^{i\ell} = \frac{Y_\ell U_\ell}{Y_i U_i} \; for \; i \in \{i, j, k\} \tag{3.3}$$

From the first order conditions (3.1) and the equilibrium on the goods and money markets (2.11) and (2.12) we get the stationary consumption levels and money demands (see Appendix A). But the actual patterns of exchange are still to be specified.

[18] We will show below under which condition this is true.
[19] U_i, U_j, U_k, Z_{ψ} are partial derivatives.
[20] The nominal exchange rates are therefore given by:

$$\tilde{e}^{i\ell} = \frac{Y_\ell U_\ell M_i}{Y_i U_i M_\ell} \text{ for } i \in \{i, j, k\}$$

3.3. Patterns of currency exchange

Note that although a transaction involves two parties, it only contributes once to liquidity. So if we call $V_{ij,t}$ the double of the total volume exchanged on bilateral foreign exchange market ij, (2.9) and (2.14) give:

$$\sum_{\ell} s_{ij}(\ell) = \sum_{\ell} v_{ij}(\ell) = L_{ij}^{\alpha+1} = V_{ij} \text{ for ij} \in \{ij, jk, ik\}$$

The amount of labour used in transactions in country ℓ on market ij is therefore $l_{ij}^{FI}(\ell) = \frac{v_{ij}(\ell)}{(V_{ij})^{\frac{\alpha}{\alpha+1}}}$ where $v_{ij}(\ell)$ is the volume actually exchanged on market ij by country ℓ.
The unit cost of a transaction service for country ℓ on market ij is:

$$T_{ij}(\ell) = w_{\ell} L_{ij}^{-\alpha} = \frac{w_{\ell}}{(V_{ij})^{\frac{\alpha}{\alpha+1}}} \tag{3.4}$$

where w_{ℓ} is the normalised wage in country ℓ. The cost of a transaction service $T_{ij}(\ell)$ is decreasing with the global volume actually exchanged on the market ij. The stronger the positive externality - the bigger is α - the bigger the decrease in cost when the volume increases. The equilibrium wage w_{ℓ} is the same for producers and intermediaries within country ℓ and is equal to p_{ℓ}. The equilibrium revenue R_{ℓ} of producers and intermediaries is given by $R_{\ell} = \frac{p_{\ell}Y_{\ell}}{\Lambda_{\ell}}$. Note that equilibrium wage and revenue differ since producers first get their wages, out of which they pay for the transaction service. As intermediaries are also consumers, they also pay themselves for the transaction service.

Let us take an example to illustrate the relation between the transaction costs and the structure of currency exchange. The choice of currency i as vehicle currency in a structure of partial indirect exchange is consistent with the derived transaction costs if the two following conditions are satisfied:

$$(1)\ T_{ij} \leq T_{jk};\ T_{ik} \leq T_{jk} \text{ and } (2)\ T_{ij} + T_{ik} > T_{jk}$$

Condition (1) states that currency i is the "cheapest" currency to use for indirect exchange, and condition (2) ensures that the assumed structure of partial indirect exchange is not more costly than a structure of total indirect exchange, where i is vehicle currency. If either (1) or (2) is not satisfied, then our assumption on the structure of exchange is violated for the range of parameters considered.

3.4. Utility function and interpretation of parameters

To gain in clarity without losing much in terms of generality, we restrict further the parameters of the utility function. From now on, we assume the following specification for the utility functions:

$$\begin{aligned} U^i(C_i, C_j, C_k) &= (1-2\omega)\ln(C_i^i) + \omega\ln(C_j^i) + \omega\ln(C_k^i) \\ U^j(C_i, C_j, C_k) &= \omega\ln(C_i^j) + (1-(1+\tau)\omega)\ln(C_j^j) + \tau\omega\ln(C_k^j) \\ U^k(C_i, C_j, C_k) &= \omega\ln(C_i^k) + \upsilon\omega\ln(C_j^k) + (1-(1+\upsilon)\omega)\ln(C_k^k) \end{aligned}$$

which can be summarized in the following table:

	good i	good j	good k
country I	1-2ω	ω	ω
country J	ω	1-(1+τ)ω	$\tau\omega$
country K	ω	$\upsilon\omega$	1-(1+υ)ω

The domain of variation of ω is $\left[0, \frac{1}{2}\right]$ and $(\tau, \upsilon) \in \left[0, \frac{1}{\omega} - 1\right] \times \left[0, \frac{1}{\omega} - 1\right]$. The parameters of the utility function will be interpreted as "net" or "compound" parameters describing both preferences and utility costs of the trading process between the different countries[21].

In this perspective, ω, τ and υ are three parameters which can be interpreted as describing the degree of integration of the different countries. ω may be seen as a coefficient of openness of the countries and as an indicator of global trade bariers or transportation technologies. An increase in ω can be interpreted as a global decrease in transport and trade costs leading to a global increase in international exchange. τ and υ describe the integration of country J with country K. If τ is big for example, it suggests that the good of country K is highly desired by country J and that the utility cost of acquiring it is low[22]. Characterising the domains of existence of the equilibria as a function of the trade parameters is the subject of the next section.

3.5. Existence and description of equilibria

Proposition 1. *There are six types of equilibria: three partial indirect exchange equilibria (each of the three currencies can be the vehicle) and three total indirect*

[21] For example if c is a trading cost proportional to the quantity demanded, $\omega' = \frac{\omega}{1+c}$ can be interpreted as a "net" utility parameter.

[22] Note that if we had chosen completely symmetric utility functions, the issue of vehicle currency would have lost interest since trade flows would have been bilaterally balanced.

exchange equilibria (each of the three currencies can be the vehicle).
Proof: see Appendix B.

Equilibrium E1: Partial indirect exchange equilibrium with currency i as the vehicle currency

Transaction costs are consistent with the choice of i as a vehicle currency and with a structure of partial indirect exchange if and only if:

$$(1)\ T_{ij} \leq T_{jk} \text{ and } T_{ik} \leq T_{jk};\ (2)\ T_{ij} + T_{ik} \geq T_{jk}$$

From the money demands, we can deduce that the domain of existence of E1 is the set of (τ, υ) defined by[23]:

$\{\upsilon \leq \tau \text{ and } H(\upsilon) \leq \tau \leq \frac{1}{2\upsilon-1}\} \cup \{\upsilon \geq \tau \text{ and } H'(\upsilon) \leq \tau \leq \frac{1}{2} + \frac{1}{2\upsilon}\}$ with $H(\upsilon) = \frac{h_1(\upsilon)-\upsilon-1}{1-2h_1(\upsilon)}$, $h_1(\upsilon) = \frac{\upsilon}{\left(1-\upsilon^{\frac{\alpha}{\alpha+1}}\right)^{\frac{\alpha+1}{\alpha}}}$. $H'(\upsilon)$ is the symmetric of $H(\upsilon)$ with respect to the 45-degree line. E1 has been represented in graph 4a (hatched region). The graph shows that this structure of exchange exists when countries J and K "do not trade too much with each other". It is possible for country J (respectively country K) to have a high demand for the good of country K (respectively country J). Currency i can be a vehicle currency, however, only when trade flows between country J and K are far from being bilaterally balanced. Furthermore, if trade flows between countries J and K are relatively small compared to trade flows between these countries and country I, then currency i will be the vehicle currency in all cases. The intuition is simple: if countries J and K have very strong trade links, they find it more advantageous to use one of their currencies instead of using the currency of country I. Conversely, if country I is very open then currency i is very likely to be the vehicle currency. In the limiting case where $\omega = 1/2$, the domain of existence of equilibrium E1 is the largest possible domain, since τ and υ are constrained to belong to $[0, 1/\omega - 1]$=$[0, 1]$.

We have seen the conditions on trade flows that render the use of currency i as a vehicle currency possible. But we need to specify further the domain of existence of equilibrium E1 and to rule out the possibility of total indirect exchange through currency i. Total indirect exchange can be shown to occur whenever countries J and K do not trade very much with each other or if their trading relations are very asymmetric, country J desiring highly the good of country K, for example, but facing a very low demand from country K. It can be shown that the leftwards frontier of our domain shifts to the right when the thick market

[23] See appendix B.1 for the analytical derivation of the boundaries.

externality increases: the stronger the externality, the less costly is total indirect exchange, where all currency exchange processes occur on two markets instead of three, relative to partial indirect exchange.

Equilibrium E2: Partial indirect exchange equilibrium with currency j as the vehicle currency

Transaction costs are consistent with the choice of j as vehicle currency if and only if:

$$1)\ T_{ij} \leq T_{ik} \text{ and } T_{jk} \leq T_{ik};\ 2)\ T_{ij} + T_{jk} \geq T_{ik}$$

The domain of existence of E2 is the set of (τ, υ) defined by[24]:
$\left\{\upsilon \leq \tau;\ \tau \geq \frac{1}{2} + \frac{1}{2\upsilon}\right\} \cup \{\tau \leq \upsilon; \upsilon \geq 1;\ \tau \geq h_2(\upsilon)\}$
with $h_2(\upsilon) = \frac{1}{2}\left(1 - \upsilon^{\frac{\alpha}{\alpha+1}}\right)^{\frac{\alpha+1}{\alpha}} (2\upsilon + 1) - 1/2$

E2 has been represented in graph 4b (hatched area). This equilibrium exists only when countries J and K trade sufficiently with each other or if the good of country J is highly desired. It is apparent that the use of a currency as vehicle swells the foreign exchange markets in that currency. For example, even for values of the trade parameters for which one would have intuitively expected currency k to become vehicle currency - when the good of country K is highly desired and/or trading costs with country K are very low - equilibrium E2 exists. We are in the presence of a *lock-in effect*. In the limiting case of complete openness of country I, however, equilibrium E2 does not exist. For very high levels of demand for the good of country J, equilibrium E2 does not exist either, since total indirect exchange is cheaper. It can be shown that the righthand frontier of the domain shifts to the left when the thick market externality increases. Total indirect exchange thus becomes more attractive when the externality is powerful.

Equilibrium E3: Partial indirect exchange equilibrium with currency k as vehicle currency.

The domain of existence of this equilibrium is symmetric to the domain of existence of E2 with respect to $\tau = \upsilon$.

Equilibrium E4 (resp. E5, E6): Total indirect exchange equilibria with currency i (resp. j, k) as vehicle currency.

Transaction costs are consistent with the choice of i (resp. j, k) as vehicle currency if and only if:

[24] See appendix B.2 for the analytical derivation of the boundaries.

$$T_{ij} + T_{ik} \leq T_{jk};\ resp.\ T_{ij} + T_{jk} \leq T_{ik}; resp.\ T_{ik} + T_{jk} \leq T_{ij}$$

These types of equilibria exist on the whole domain as long as the externality is positive. The lock-in effects are very strong, since for an individual transactor it is then infinitely costly to set up a bilateral foreign exchange market that does not exist[25].

Proposition 2. *For a given* (τ, υ, ω), *there are multiple equilibrium structures of currency exchange.*

Proof: Immediate, given the above description of the domains of existence of E1-E6.

The intuition behind the multiplicity of equilibria is the following: if trade flows are not bilaterally balanced, one of the currencies has to be used as a vehicle, which swells the foreign exchange markets in that currency. Thus transaction costs are lowered and the emergence of a currency as a vehicle is self-enforcing. Such a phenomenon can lead to a structure of partial indirect exchange or total indirect exchange for some values of the parameters. It is the double coincidence of wants problem in the currency exchange process which is at the root of the existence of vehicle currencies and of the multiplicity of the equilibria.

4. Interpretation: the rise and fall of the pound and the dollar as vehicle currencies.

Graph 5 summarises the domains of existence of all equilibria.

Call country J Britain, country K the Commonwealth (in fact many countries) and country I the USA. Start at point O on graph 5, i.e,. a situation where trading costs between Britain and the Commonwealth are not very high and the USA is fairly closed. In such a world global trade links are not very strong except between Britain and the Commonwealth. A priori the vehicle currency could be either the pound sterling or the Commonwealth currency (of which there were many). Assume that we start off with the pound as vehicle. Then we are in the case of graph 4b. If there is an exogenous decrease of the trade links between Britain and the Commonwealth (we assume here a parallel decrease in υ and τ, but

[25] This should be seen as a limiting case. It is straightforward to weaken the lock-in effect by changing slightly the functional form of the transaction technology. But this complicates things (by adding new boundaries) without providing new insights.

any decrease could be considered), e.g., because of a war, then for a certain range of the parameters the pound can remain a vehicle currency even though all three currencies could now play this role if we looked only at the "fundamentals"(point O′). This is the *hysteresis effect* in the use of an international currency that has been often described. A further decrease in τ and υ leads to a region where only the dollar or the pound are potential vehicle currencies. But, if the economy happens to be located in O′′ after the Second World War, for example, where trade links have been severely disrupted, then it becomes cheaper to use the dollar as vehicle currency instead of the pound. This leads to a change in the structure of exchange, and the economy has now the configuration of graph 4a, where the dollar has supplanted the pound as vehicle currency.

Note that the model predicts that the pattern of exchange is determined by the trade links between the different countries and *not* by the relative sizes of the different economies. Table 1 shows that the GDP of the USA was already bigger than the GDP of the UK in 1870. It was a bit less than double the GDP of the UK in 1900, when the supremacy of the British pound as international currency was uncontested. UK merchandise exports were greater (in value) than US merchandise exports until the Great Depression; at that time the value of US merchandise exports represented approximately 3% of US GDP while the value of UK merchandise exports accounted for roughly 13% of UK GDP. Only after World War II did the exports of the US overtake significantly the exports of the UK. These data are consistent with our result that trade flows are the key determinants in the process of internationalisation of currencies, as opposed to economic size[26].

Of course the process described here is highly simplified[27], but it seems to give some valuable insights about the life - and death - of international currencies.

5. Structures of exchange and real output loss

Proposition 3. *For given trade parameters (τ,υ,ω) there is a unique equilibrium structure of exchange that minimises world output loss. In particular, if the thick market externality is greater than a threshold value $\bar{\alpha}$, total indirect exchange in one of the three currencies always minimises world output loss compared to partial indirect exchange in that currency.*
Proof: See Appendix C.

[26] See section 6 for a discussion of this result.
[27] In particular, it relies on steady-state analysis.

The results are portrayed in graphs 6a and 6b. In our model, real transaction costs occur through the allocation of an endogenous amount of productive labour of each country into the transaction sector. The bigger this amount, the higher the output loss. The global amount of labour involved in the financial intermediation sector in the world is:

$$L_{FI} = \sum_{\ell} \sum_{ij} \frac{v_{ij}(\ell)}{(V_{ij})^{\frac{\alpha}{\alpha+1}}} = \sum_{ij} (V_{ij})^{\frac{1}{\alpha+1}} \tag{5.1}$$

The key point here is that different exchange structures lead to different V_{ij} which has a repercussion on L_{FI}. The magnitude of this repercussion depends on the strength of the externality. It is found (see Appendix C) that if the externality is greater than $\bar{\alpha}$, then the structures of exchange which minimise world output loss are always totally indirect, which currency being used depending on the trade parameters (graph 6a). $\bar{\alpha}$ is the value above which, in a totally symmetric environment ($\omega = \tau\omega = \upsilon\omega = 1/3$), it is less costly to switch to a structure of total indirect exchange even though all the foreign exchange markets are bilaterally balanced. Below $\bar{\alpha}$ the structures of exchange which minimise output loss can either be partially indirect or totally indirect in either of the three currencies depending on the trade parameters (graphs 6a and 6b).

When trade flows are very symmetric, partial indirect exchange tends to be cheaper. But when trade flows are very asymmetric, total indirect exchange is less costly in general. The areas where some kind of total indirect exchange minimises output loss shrink when the externality decreases, and conversely. In the limiting case where there is no externality, the less costly structure of exchange is one of partial indirect exchange, and which currency is used as vehicle is irrelevant. Some indirect exchange is needed to clear the foreign exchange markets anyway, but since there is no thick market externality, any of the currencies qualifies. A structure of partial indirect exchange minimizes the amount of money indirectly exchanged and therefore has lower costs than a structure of total indirect exchange.

In the other limiting case where the externality is infinite, the optimal structure of exchange is one of total indirect exchange, and which currency is used as vehicle is again irrelevant. Here the thick market externality is so strong that the underlying trade pattern does not matter any more, the only requirement being to "pool" the volumes of currency exchange as much as possible. In our three-country world, this leads to the disappearance of one of the three foreign exchange markets. For intermediate values of the externality (below $\bar{\alpha}$), the less costly structure of exchange results from the balance of two factors: the strength of the externality and the underlying structure of payments.

6. Modelling strategy

In this section we discuss three issues: the modelling of the cash-in-advance constraint, the form of the utility function and potential micro-foundations for the thick market externality.

Cash-in-advance constraints

We assumed that all goods purchases have to be paid for in the currency of the seller. This is a less than satisfactory assumption for a general equilibrium model. Ideally[28] the model should impose a generic cash-in-advance constraint and not a country-specific one. But this would effectively come down to writing a theory of trade invoicing, which unfortunately does not exist so far. To go beyond a highly restrictive formulation[29], we would have to introduce additional features. If the internationally traded net production resulted from a production technology involving lots of transactions in domestic inputs, for example, then it could be rational to invoice the final good in domestic currency, in order to avoid transaction costs. This would however complicate the model considerably. And in reality, the majority of the trade occurring among developed countries is paid for in the currency of the exporter[30].

Form of the utility function

As mentioned earlier, the model predicts that the pattern of exchange is determined by the trade links between the different countries and *not* by the relative sizes of the different economies. This strong result should be regarded as a benchmark case. With a Cobb-Douglas utility function, changes in output are fully compensated by changes in the terms of trade, which keeps constant *in value* the amount of currencies traded. In a more general specification (CES), both size and trade links would matter. We would get the intuitive result that the likelihood of

[28] As one referee pointed out clearly.

[29] For a two-country model with a generic cash-in-advance constraint but set up in a much simpler economic environment, see Matsui [1998]. In that paper agents always use the currency which is the less inflationary, unless there are specific exogenous requirements (like paying taxes in domestic currency).

[30] Except for Japan. In 1992, the shares of exports invoiced in domestic currency were 92% for the US, 77% for Germany, 62% for the UK, 55% for France, 40% for Japan. For more detailed studies of the invoicing patterns see Grassman [1973] and McKinnon [1979].

a currency becoming the vehicle currency increases with the size of the country issuing it, *ceteris paribus*[31].

Microfounding the thick market externality
We could develop a search model à-la-Diamond (1984) on each of the bilateral foreign exchange markets. Such a model, where the probability of finding a trading partner is increasing in the number of potential traders, would generate an increasing return to scale transaction technology of the form assumed.

7. Conclusion

The paper has developed a three-country, three-currency model of the world economy to study the emergence of vehicle currencies. A "thick" market externality is assumed on each bilateral foreign exchange market: such an assumption is supported by systematic empirical work. There are multiple equilibria, due to the absence of "double coincidence of wants" on those markets and to complementarity in the strategy of the agents. We have characterized the equilibria using the underlying trade parameters and the strength of the externality as key variables. We have applied this analysis to explain the rise of the dollar and the fall of the pound as international media of exchange. By comparing the different equilibria, we have also been able to analyse the impact of the different equilibrium exchange structures on world output.

It would be heroic to pretend that changes in trade flows are the only determinants of currency internationalisation[32]. In our example, the relative roles of Britain and the US in world financial trade, the relative commitments to remain (or return to) convertibility with gold, the relative riskiness of the pound and the dollar were also undoubtedly important[33]. It is extremely difficult, however, to disentangle the impacts of those factors. It is not unreasonable to think that trade in short-term financial assets could be modelled along the lines developed here: liquidity matters for markets where investors park their funds temporarily[34]. But foreign direct investment and other types of long-term capital flows, as well as credibility issues, are clearly beyond the scope of this paper.

[31] But we could not solve analytically for all the results. See Rey (1998).

[32] Though the impact of trade has been emphasised by several authors (see Cohen 1971)

[33] I am grateful to an anonymous referee for spelling out very clearly those different factors. For detailed discussions of the evolution of the International Monetary System, see Friedman and Schwartz (1963), Eichengreen (1992).

[34] See Rey (1998) for a theoretical model involving bond trade.

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TABLE 1

Years	GDP UK	GDP USA	Merch. Exp. UK	Merch. Exp. USA
1820	**34829**	12432	**1125**	251
1850	**60479**	42475		
1870	95651	**98418**	**12237**	2495
1900	176504	**312866**		
1913	214464	**517990**	**39348**	19196
1929	239985	**844324**	**31990**	30368
1950	344859	**1457624**	39348	**43114**
1973	674061	**3519224**	94670	**174548**
1992	910401	**5510378**	194535	**451026**

All the figures are in million 1990 dollars. Source: Maddison, A., *Monitoring the world economy 1820-1992*, Paris OECD.

STRUCTURES OF EXCHANGE

Figure 1: Aggregate Balance of Payment Equilibrium for three countries

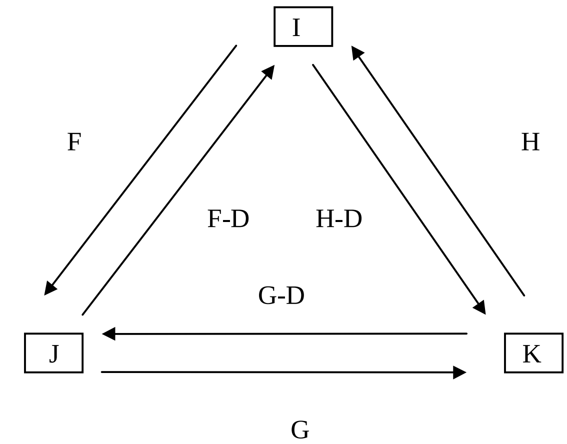


Figure 2: Partial Indirect Exchange

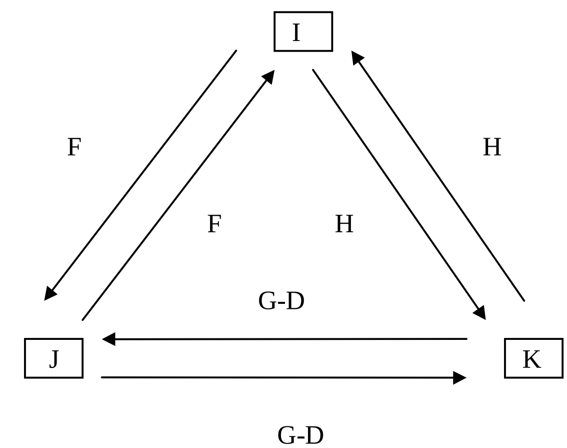
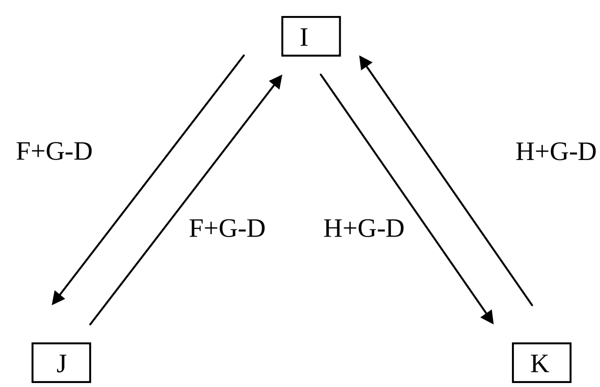


Figure 3: Total Indirect Exchange



EXISTENCE OF EQUILIBRIA

Figure 4a: Partial Indirect Exchange and Currency i is Vehicle.

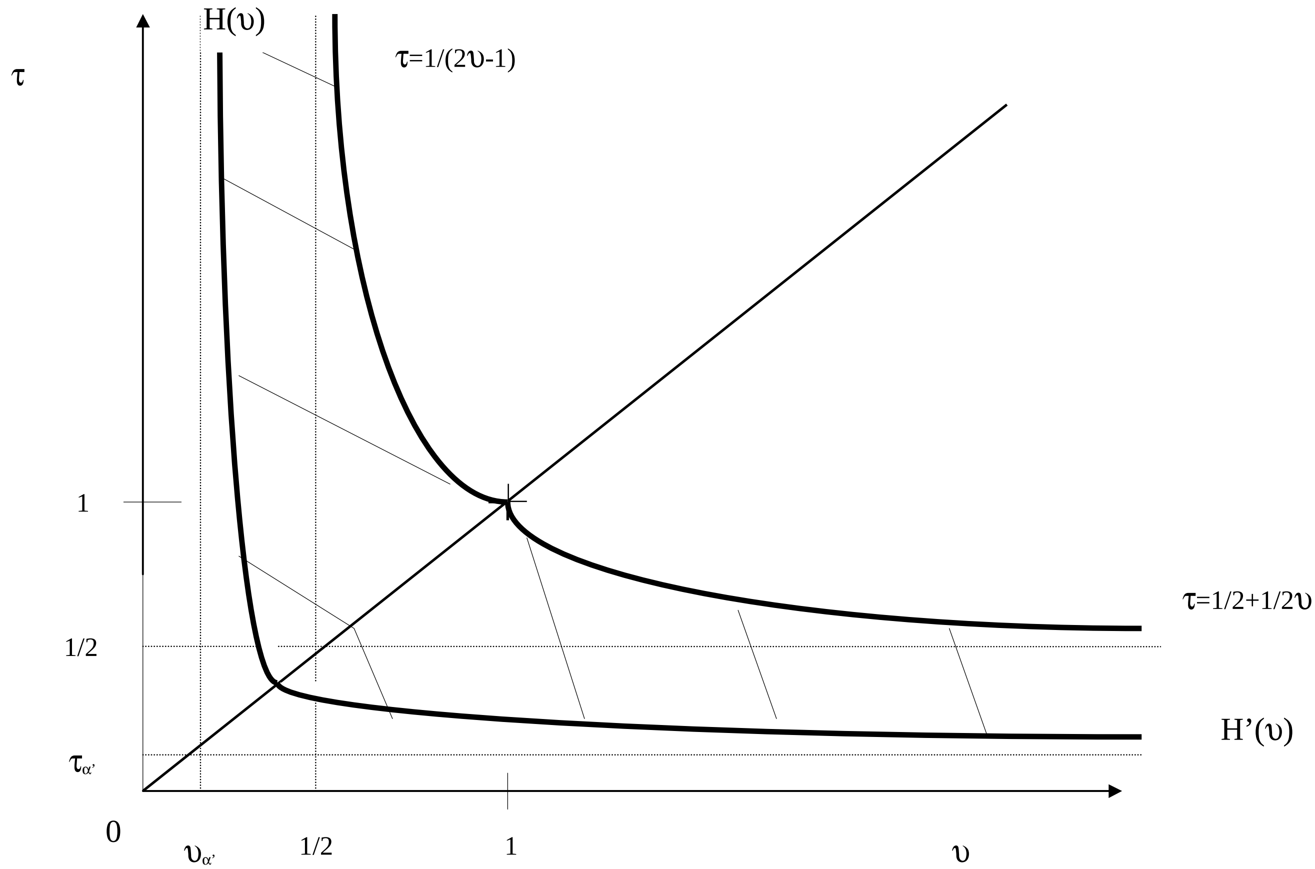


Figure 4b: Partial Indirect Exchange and Currency j is Vehicle.

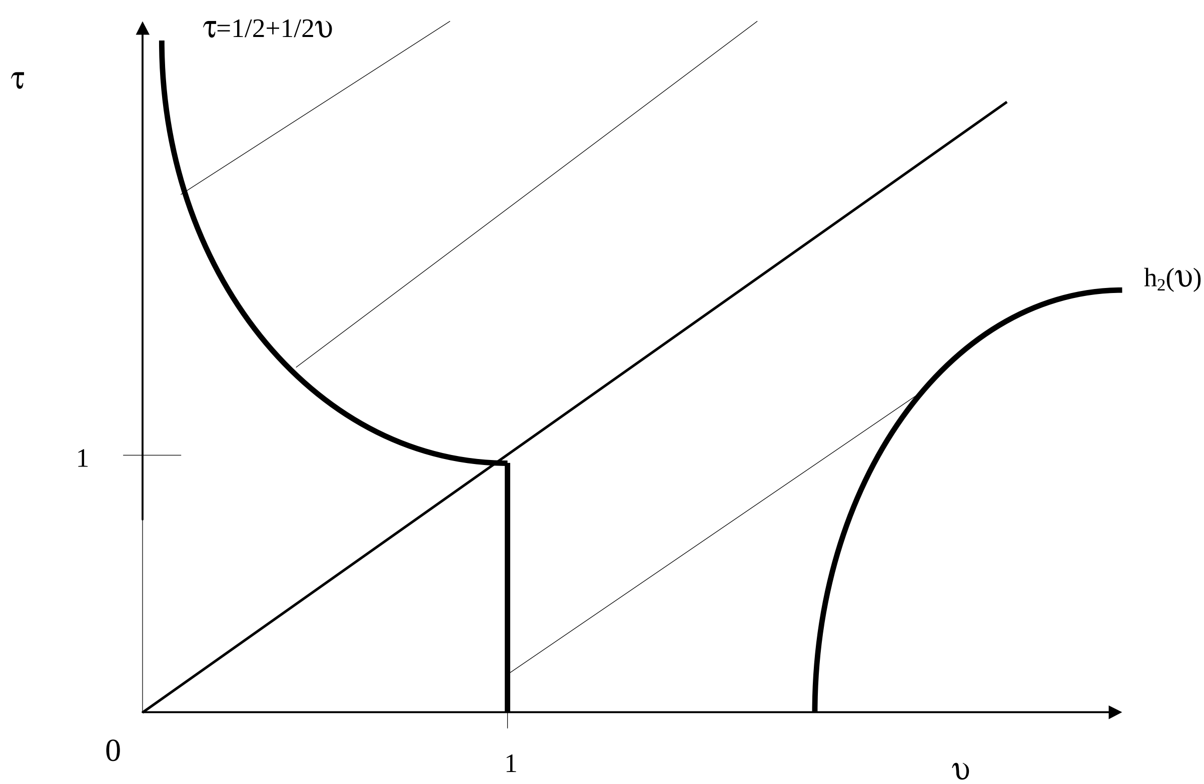
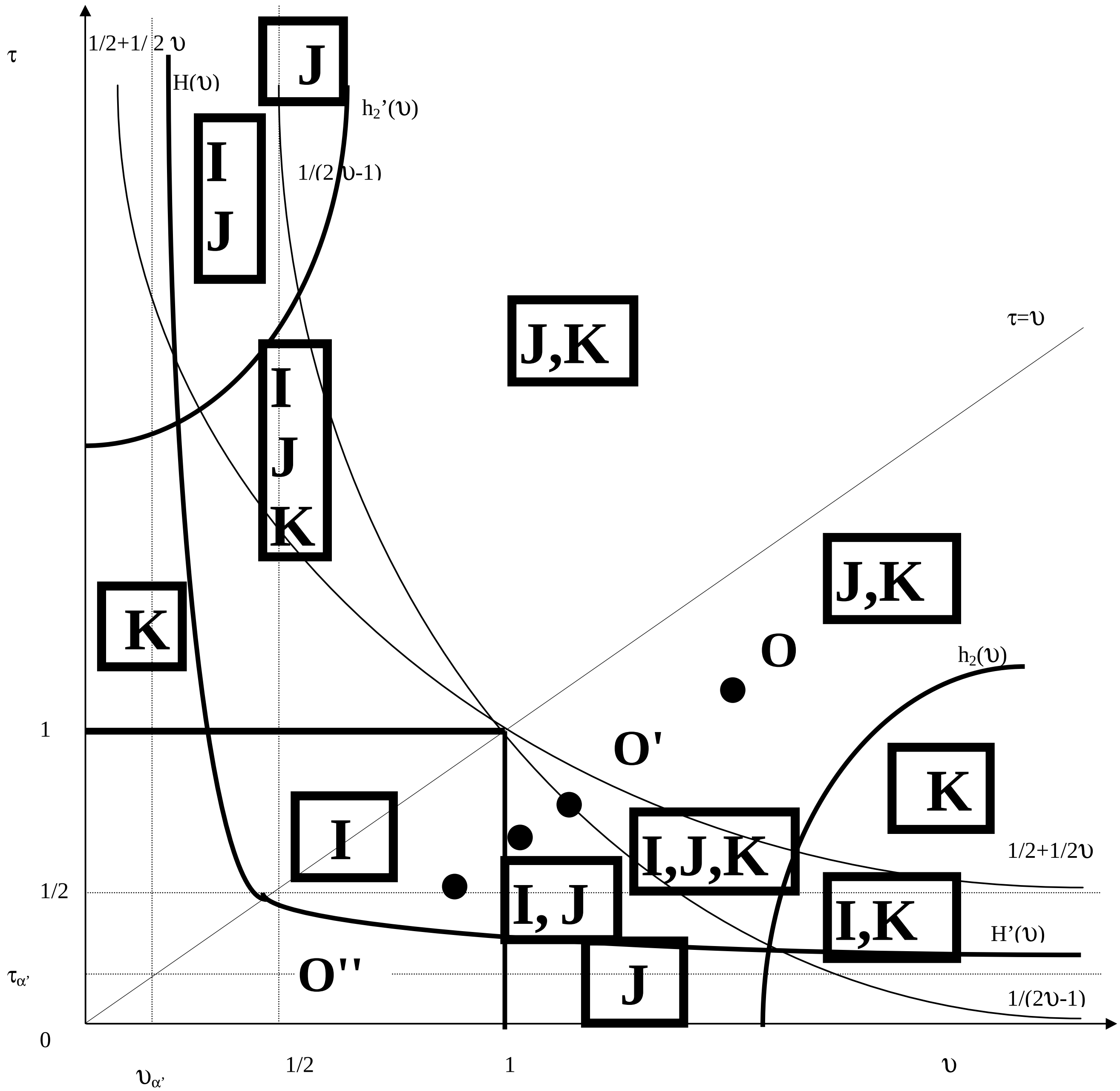


Figure 5: Multiple Equilibria.

Application to the British Pound and the US Dollar.



Output Loss

Figure 6a: Strong externality.

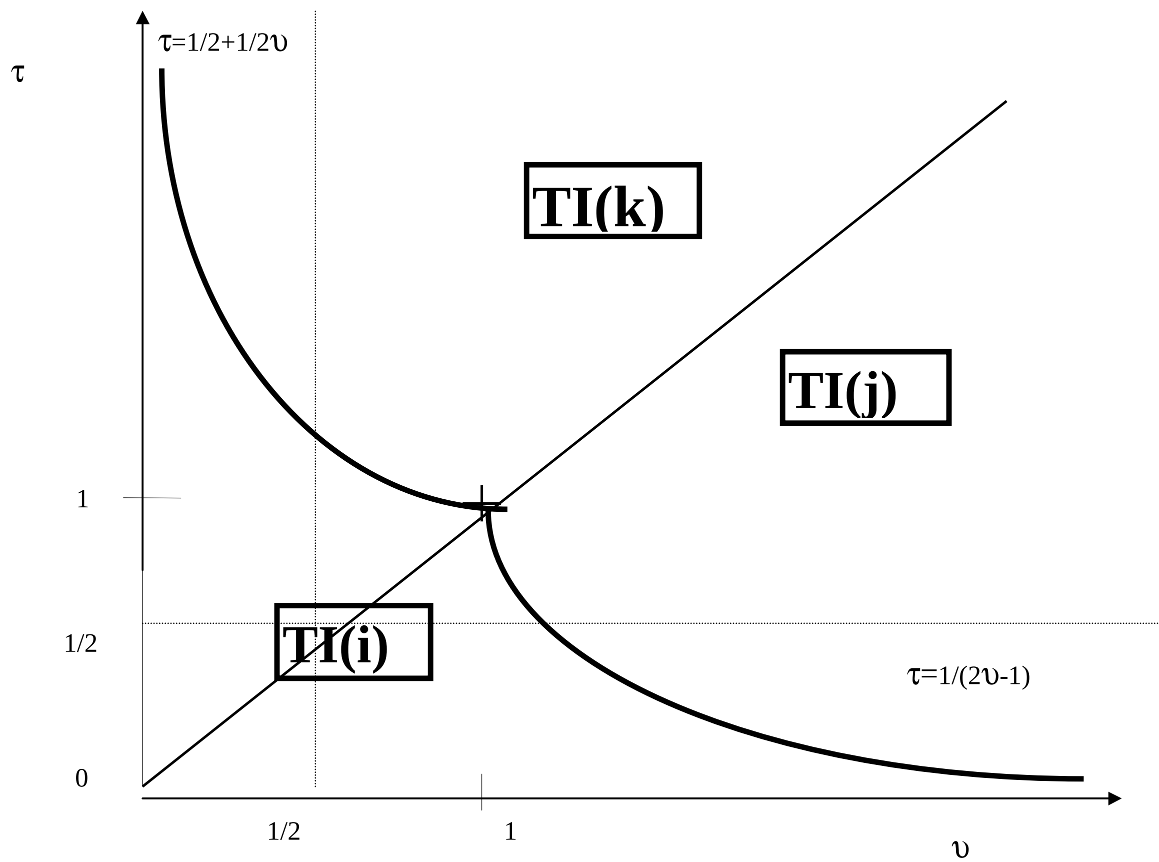
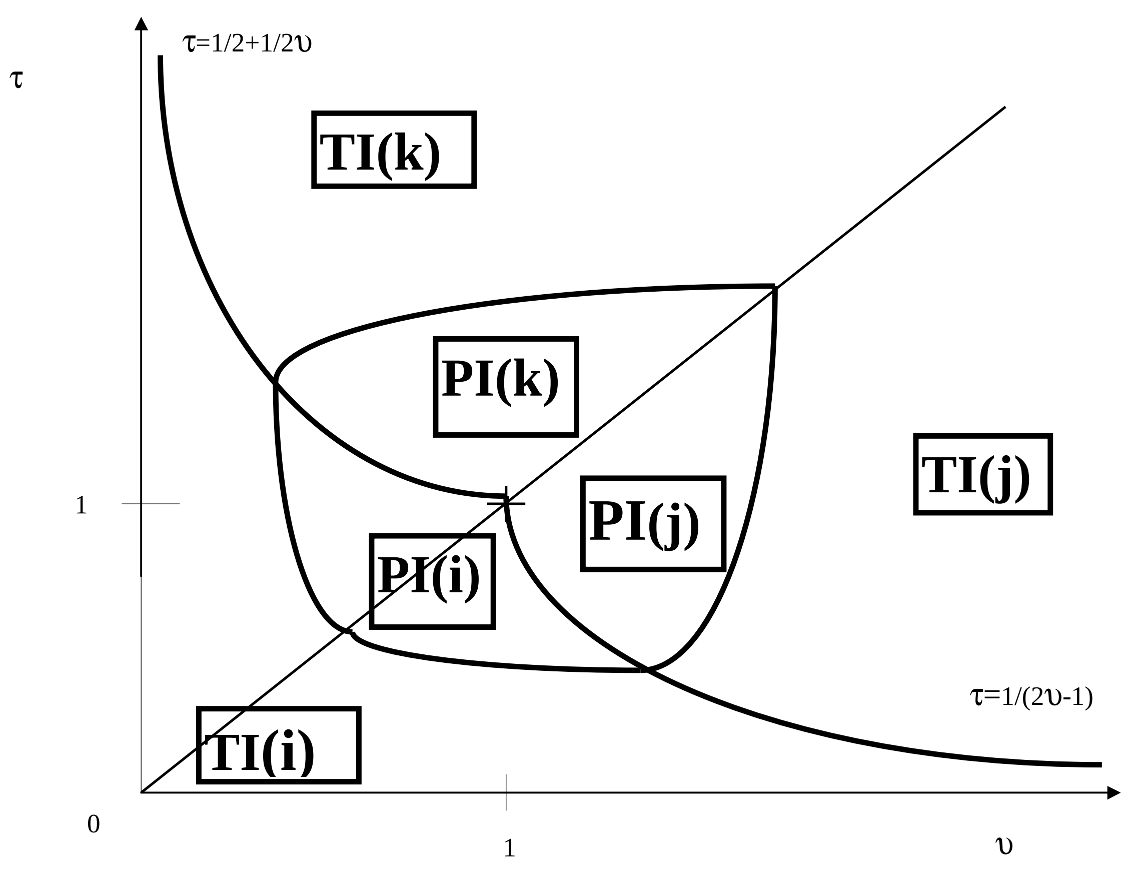


Figure 6b: Weak externality.



A. Steady state exchange rates and consumption levels.

From the first order conditions and the goods market equilibria we can derive the steady state exchange rates:

$$e_{ij} = \frac{\left(\alpha_j^i \alpha_i^k + \alpha_j^k \left(1 - \alpha_i^i\right)\right)}{\left(\alpha_i^j \alpha_j^k + (1 - \alpha_j^j)\alpha_i^k\right)};\ e_{ik} = \frac{\left((1 - \alpha_i^i)\,\alpha_k^j + \alpha_i^j \alpha_k^i\right)}{\left(\alpha_i^j \alpha_j^k + (1 - \alpha_j^j)\alpha_i^k\right)}$$

We can also derive the consumption levels[35] :

$$C_i^i = \alpha_i^i Y_i;\ C_j^i = \alpha_j^i \frac{\left(\left(1 - \alpha_j^j\right) \alpha_i^k + \alpha_i^j \alpha_j^k\right)}{\left((1 - \alpha_i^i)\alpha_j^k + \alpha_j^i \alpha_i^k\right)} Y_j;\ C_k^i = \alpha_k^i \frac{\left(\left(1 - \alpha_j^j\right) \alpha_i^k + \alpha_i^j \alpha_j^k\right)}{\left(\left(1 - \alpha_i^i\right) \alpha_k^j + \alpha_i^j \alpha_k^i\right)} Y_k$$

We can then derive the various money demands:

$$V_{i \to j} = p_j C_j^i$$

which means that the consumer of country i wants to exchange $V_{i \to j}$ "real units" of currency i against currency j[36].

$$V_{j \to i} = \alpha_i^j \frac{\left((1 - \alpha_i^i)\,\alpha_j^k + \alpha_j^i \alpha_i^k\right)}{\left((1 - \alpha_j^j)\alpha_i^k + \alpha_i^j \alpha_j^k\right)}\ ;\ V_{i \to j} = \alpha_j^i \frac{\left(\left(1 - \alpha_j^j\right) \alpha_i^k + \alpha_i^j \alpha_j^k\right)}{\left((1 - \alpha_i^i)\alpha_j^k + \alpha_j^i \alpha_i^k\right)}$$

B. Domain of existence of the equilibria

To derive the domains of existence of the different equilibria requires:
1) computing the volumes actually exchanged on each bilateral foreign exchange market under the assumption that one of the six possible structures of exchange is adopted;
2) checking that equilibrium transaction costs support the structure of exchange that has been assumed.

B.1. Partial indirect exchange with currency i as the vehicle: E1

We prove here that the domain of existence of E1 is the set of (τ, υ) defined by:

[35] The other results can be obtained by permutation of the indices $\{i, j, k\}$.
[36] The other results can be obtained by permutation of the indices $\{i, j, k\}$.

$\left\{\upsilon \leq \tau \text{ and } H(\upsilon) \leq \tau \leq \frac{1}{2\upsilon-1}\right\} \cup \left\{\upsilon \geq \tau \text{ and } H'(\upsilon) \leq \tau \leq \frac{1}{2}+\frac{1}{2\upsilon}\right\}$ with $H(\upsilon) = \frac{h_1(\upsilon)-\upsilon-1}{1-2h_1(\upsilon)}$, $h_1(\upsilon) = \frac{\upsilon}{\left(1-\upsilon^{\frac{\alpha}{\alpha+1}}\right)^{\frac{\alpha+1}{\alpha}}}$; $H'(\upsilon)$ is the symmetric of $H(\upsilon)$ with respect to the 45-degree line.

1) The actual volumes exchanged on the different markets (expressed in normalised units of currency i) are:

$$\begin{aligned} V_{ij} &= \omega \max\left(1, \frac{1+2\upsilon}{\tau+1+\upsilon}\right);\ V_{ik} = \omega \max\left(1, \frac{1+2\tau}{\tau+1+\upsilon}\right) \\ V_{jk} &= \omega \min\left(\frac{\tau(1+2\upsilon)}{\tau+1+\upsilon}, \frac{\upsilon(1+2\tau)}{\tau+1+\upsilon}\right) \end{aligned}$$

2) E1 is an equilibrium iff $T_{jk} \geq T_{ij}$, $T_{jk} \geq T_{ik}$ and $T_{ij}+T_{ik}-T_{jk} \geq 0$.

•$T_{jk} \geq T_{ij}$ and $T_{jk} \geq T_{ik} \Leftrightarrow \left\{\tau \geq \upsilon \text{ and } \tau \leq \frac{1}{2\upsilon-1}\right\} \cup \left\{\tau \leq \upsilon \text{ and } \tau \leq \frac{1}{2\upsilon}+\frac{1}{2}\right\}$

• We interpret $T_{ij}+T_{ik}-T_{jk}$ as a function of τ, $f(\tau)$, for all values of υ. If $\tau \geq \upsilon$, we have to consider only $\upsilon \in [0,1]$; otherwise, we are not in equilibrium. We can show that $f(\tau)$ is increasing $\forall \tau \in \left[\upsilon, \frac{1}{2\upsilon-1}\right]$ if $\upsilon \geq 1/2$ and $\forall \tau \in [\upsilon, \infty]$ if $\upsilon \leq 1/2$.

$f(\tau=\upsilon) \geq 0 \Longleftrightarrow \upsilon \geq \left(\frac{1}{2}\right)^{\frac{1+\alpha}{\alpha}}$, which we will denote by υ_α.

• if $\upsilon \geq \upsilon_\alpha$ then $f(\tau) \geq 0\ \forall \tau \in \left[\upsilon, \frac{1}{2\upsilon-1}\right]$ if $\upsilon \geq 1/2$ and $\forall \tau \in [\upsilon, +\infty]$ if $\upsilon \leq 1/2$.

• if $\upsilon \leq \upsilon_\alpha$ then $f(\tau=\upsilon) \leq 0$; $\lim_{\tau\to\infty} f(\tau) \geq 0 \Leftrightarrow \upsilon \geq \left(\frac{1}{\left(1+2^{\frac{\alpha}{1+\alpha}}\right)}\right)^{\frac{1+\alpha}{\alpha}} = \upsilon_{\alpha'}$

In summary:

• if $\upsilon \geq 1/2$ then $f(\tau) \geq 0$

• if $\upsilon \leq \upsilon_{\alpha'} \leq \upsilon_\alpha \leq 1/2$ then $f(\tau) \leq 0\ \forall \tau \in [\upsilon, \infty]$ (since $f(\tau=\upsilon) \leq 0$ and $\lim_{\tau\to\infty} f(\tau) \leq 0$ and f is increasing)

• if $\upsilon_{\alpha'} \leq \upsilon \leq \upsilon_\alpha \leq 1/2$ then $\exists!\ \tau*$ such that $f(\tau*) = 0$ with $\tau* \in [\upsilon, \infty]$; $\tau* = \frac{h_1(\upsilon)-\upsilon-1}{1-2h_1(\upsilon)} = H(\upsilon)$ where $h_1(\upsilon) = \frac{\upsilon}{\left(1-\upsilon^{\frac{\alpha}{1+\alpha}}\right)^{\frac{1+\alpha}{\alpha}}}$

If $\upsilon \leq 1/2$, $f(\tau) \geq 0 \Longleftrightarrow \tau \geq H(\upsilon)$

$H(\upsilon)$ admits $\upsilon = \upsilon_{\alpha'}$ for asymptote and intersects $\tau = \upsilon$ in υ=υ_α.

We deduce the case $\tau \leq \upsilon$ by symmetry with respect to the 45 degree line.

Q.E.D.

All the results are summarised on graph 4a.

B.2. Partial indirect exchange with currency j as the vehicle: E2

We prove here that the domain of existence of E2 is the set of (τ, υ) defined by: $\left\{\upsilon \leq \tau;\ \tau \geq \frac{1}{2} + \frac{1}{2\upsilon}\right\} \cup \{\tau \leq \upsilon; \upsilon \geq 1;\ \tau \geq h_2(\upsilon)\}$
with $h_2(\upsilon) = \frac{1}{2}\left(1 - \upsilon^{\frac{\alpha}{\alpha+1}}\right)^{\frac{\alpha+1}{\alpha}}(2\upsilon+1) - 1/2$
1) The actual volumes exchanged (expressed in normalised units of currency i) are now:

$$V_{ij} = \omega \max\left(1, \frac{1+2\upsilon}{\tau+1+\upsilon}\right); V_{ik} = \omega \min\left(1, \frac{1+2\tau}{\tau+1+\upsilon}\right)$$
$$V_{kj} = \omega \max\left(\frac{\tau(2\upsilon+1)}{\tau+1+\upsilon}, \frac{(2\tau+1)\upsilon}{\tau+1+\upsilon}\right)$$

2) E2 is an equilibrium iff $T_{ik} \geq T_{ij}$, $T_{ik} \geq T_{jk}$ and $T_{ij} + T_{jk} - T_{ik} \geq 0$.
•$T_{ik} \geq T_{ij}$, $T_{ik} \geq T_{jk} \Leftrightarrow (\tau, \upsilon) \in \left\{\upsilon \leq \tau \text{ and } \tau \geq \frac{1}{2} + \frac{1}{2\upsilon}\right\} \cup \left\{\tau \leq \upsilon; \tau \geq \frac{1}{2\upsilon} + \frac{1}{2}\right\}$
•If $\tau \geq \upsilon$, we need to have $\tau \geq \frac{1}{2\upsilon} + \frac{1}{2}$, otherwise we are not in equilibrium. Then we have $T_{ij} + T_{jk} - T_{ik} \geq 0\ \forall\tau$.
• If $\tau \leq \upsilon$, we need $\upsilon \geq 1$, otherwise we are not in equilibrium. Define $f(\upsilon, \tau) = T_{ij} + T_{jk} - T_{ik}$.
$f(\tau, \upsilon) \leq 0 \Leftrightarrow (2\tau+1) - \left(1 - \frac{1}{\upsilon^{\frac{\alpha}{\alpha+1}}}\right)^{\frac{\alpha+1}{\alpha}}(1+2\upsilon) \leq 0 \Leftrightarrow \tau \leq h_2(\upsilon)$, where $h_2(\upsilon) = \left(\upsilon^{\frac{\alpha}{\alpha+1}} - 1\right)^{\frac{\alpha+1}{\alpha}}\left(\frac{1}{2\upsilon} + 1\right) - \frac{1}{2}$.
It is easy to show that there exists a unique υ_α such that $h(\upsilon_\alpha) = 0$ as soon as $\alpha > 0$.
$\forall\upsilon$, $h_2(\upsilon) \leq \upsilon$ and $h_2(\upsilon)$ is increasing with α. It is also possible to show that in the neighbourhood of $+\infty$, $h_2(\upsilon) = \upsilon - \frac{\alpha+1}{\alpha}\upsilon^{\frac{1}{\alpha+1}} + o\left(\upsilon^{\frac{1}{\alpha+1}}\right)$

Q.E.D.

All these results are summarized on graph 4b.

B.3. Partial indirect exchange with currency k as the vehicle: E3

This case is completely symmetric to the former case, and all the results can be deduced by symmetry with respect to $\tau = \upsilon$.

B.4. Total indirect exchange with currency i as the vehicle: E4

1) $V_{ij} = \omega \frac{(1+2\upsilon)(1+\tau)}{\tau+1+\upsilon}$; $V_{ik} = \omega \frac{(1+\upsilon)(1+2\tau)}{\tau+1+\upsilon}$; $V_{jk} = 0$
2) Therefore, we always have $T_{ij} + T_{ik} \leq T_{jk}$. The whole space is an equilibrium. Results are similar for total indirect exchange structures with currency j and k as vehicles.

C. Output and structures of exchange

In this appendix, we compare the different levels of world output for different patterns of exchange.
Structures of partial indirect exchange:
When currency i is the vehicle currency, real output loss is

$$\omega \left(\begin{array}{c} \left[\max\left(1, \frac{1+2\upsilon}{\tau+1+\upsilon}\right)\right]^{\frac{1}{1+\alpha}} + \left[\max\left(1, \frac{1+2\tau}{\tau+1+\upsilon}\right)\right]^{\frac{1}{1+\alpha}} \\ + \left[\min\left(\frac{\tau(1+2\upsilon)}{\tau+1+\upsilon}, \frac{\upsilon(1+2\tau)}{\tau+1+\upsilon}\right)\right]^{\frac{1}{1+\alpha}} \end{array} \right)$$

When currency j is the vehicle currency, real output loss is

$$\omega \left(\begin{array}{c} \left[\max\left(1, \frac{1+2\upsilon}{\tau+1+\upsilon}\right)\right]^{\frac{1}{1+\alpha}} + \left[\min\left(1, \frac{1+2\tau}{\tau+1+\upsilon}\right)\right]^{\frac{1}{1+\alpha}} \\ + \left[\max\left(\frac{\tau(1+2\upsilon)}{\tau+1+\upsilon}, \frac{\upsilon(1+2\tau)}{\tau+1+\upsilon}\right)\right]^{\frac{1}{1+\alpha}} \end{array} \right)$$

Structures of total indirect exchange:
When currency i is the vehicle currency, real output loss is

$$\omega \left[\left(\frac{(1+\tau)(1+2\upsilon)}{\tau+1+\upsilon} \right)^{\frac{1}{1+\alpha}} + \left(\frac{(1+2\tau)(1+\upsilon)}{\tau+1+\upsilon} \right)^{\frac{1}{1+\alpha}} \right]$$

When currency j is the vehicle currency, real output loss is

$$\omega \left[\left(\frac{1+2\tau+2\tau\upsilon+\upsilon}{\tau+1+\upsilon} \right)^{\frac{1}{1+\alpha}} + (2)^{\frac{1}{1+\alpha}} \right]$$

Results for currency k are analogous to those for currency j (symmetry with respect to the 45 degree line).

C.1. Some interesting limiting cases

If everything is symmetric ($\upsilon = \tau = 1$), then total indirect exchange minimises output loss compared to partial indirect exchange iff $\alpha \geq \frac{2 - \frac{\ln(3)}{\ln(2)}}{\frac{\ln(3)}{\ln(2)} - 1} = \bar{\alpha}$. If α is infinite, total indirect exchange with either of the currencies is the less costly strategy; if $\alpha = 0$, partial indirect exchange with either of the currencies is less costly (taking limits in the expressions above for real output losses).

C.2. Results along $\tau = \upsilon$

•Call $f(\upsilon,\alpha) = PI_j - TI_j = PI_k - TI_k$ the differential in output loss between partial indirect exchange in currency j or k and total indirect exchange in currency j or k.

Along $\tau = \upsilon$, for $\upsilon \geq 1$ (the only region where the use of currency j is an equilibrium) it is possible to show that f is increasing in υ when α is kept constant and in α when υ is kept constant. As $f(1,\alpha) \geq 0 \Leftrightarrow \alpha \geq \bar{\alpha}$, for all $\alpha \geq \bar{\alpha}$ total indirect exchange always minimises output loss. For $\alpha \leq \bar{\alpha}$ there is a frontier under which partial indirect exchange minimises output loss (since $\lim_{\upsilon\to\infty} f > 0$ and f is increasing in both arguments). The equation of this frontier is $2 + \upsilon^{\frac{1}{\alpha+1}} - (1+\upsilon)^{\frac{1}{\alpha+1}} - 2^{\frac{1}{\alpha+1}} = 0$.

•Call $g(\upsilon,\alpha) = PI_i - TI_i$ the differential in output loss between partial indirect exchange in currency i and total indirect exchange in currency i.

Along $\tau = \upsilon$, for $\upsilon \leq 1$, g is increasing in α for all $\upsilon = \tau$. g is decreasing in υ if $\upsilon \geq \frac{1}{2^{\frac{\alpha}{\alpha+1}}-1} = \upsilon_\alpha$. $g(\upsilon = 0) = 0$ and $g(\upsilon = 1) \geq 0 \Leftrightarrow \alpha \geq \bar{\alpha}$. Therefore g is increasing in υ until υ_α is reached and decreasing after. If $\alpha \geq \bar{\alpha}$ then g remains positive on the whole domain, but if $\alpha \leq \bar{\alpha}$, then $\exists!\ \upsilon_{\alpha'}$ such that total indirect exchange minimises output loss below $\upsilon_{\alpha'}$ and partial indirect exchange minimises output loss above $\upsilon_{\alpha'}$. $\upsilon_{\alpha'}$ verifies $2 + \upsilon_{\alpha'}^{\frac{1}{\alpha+1}} - 2(1+\upsilon_{\alpha'})^{\frac{1}{\alpha+1}} = 0$. The area where partial indirect exchange minimises output loss shrinks when α increases and disappears when $\alpha = +\infty$.

C.3. Results in the entire space

• For $\tau \geq \upsilon$ and $\upsilon \leq 1$ (domain when i is vehicle currency), let us define $J(\upsilon,\alpha,\tau) = PI_i - TI_i$. By taking the derivative of $J(\upsilon,\alpha,\tau)$ with respect to τ and using that $x \to x^{\frac{1}{1+\alpha}}$ is concave for all $\alpha \geq 0$, it is possible to show that $J(\upsilon,\alpha,\tau)$ is increasing in τ for all α The same proof applies by symmetry for $\tau \leq \upsilon$.

•For $\tau \geq \upsilon$ and $\upsilon \geq 1$ (domain in which k is vehicle currency), define $K(\upsilon,\alpha,\tau) = PI_k - TI_k$.

By taking the derivative of $K(\upsilon,\alpha,\tau)$ with respect to τ and using $(1+\tau)^{-\frac{\alpha}{1+\alpha}} \leq \tau^{-\frac{\alpha}{1+\alpha}}$, it is possible to show that $K(\upsilon,\alpha,\tau)$ is an increasing function of τ for all α. The same proof applies by symmetry for $\tau \leq \upsilon$ to $PI_j - TI_j$.

Therefore:

• for $\alpha \geq \bar{\alpha}$, as total indirect exchange minimises output loss on the whole line $\tau = \upsilon$, it also dominates in the whole space. If $\tau \geq \upsilon$, currency i should be used until $\tau = \frac{1}{2} + \frac{1}{2\upsilon}$ is reached. Currency k should then be used (to minimise output loss).
• for $\alpha \leq \bar{\alpha}$, if partial indirect exchange minimises output loss along $\tau = \upsilon$ then, since we know that when $\tau \to \infty$ (resp. $\tau \to 0$) the only equilibrium structure for currency k (resp. currency i) is one of total indirect exchange, there exists a frontier which depends on α above, (resp. below) which partial indirect exchange no longer minimises output loss. The same proof applies to currency j by symmetry.

Q.E.D.

C.4. Summary

We have proved that there exists a value of α which we called $\bar{\alpha}$ such that above this value, total indirect exchange always minimises output loss compared to partial indirect exchange. Which currency should be used to minimise output loss depends on the (τ, υ).

For smaller values of α, we have proved that partial indirect exchange or total indirect exchange may minimise output loss, depending on the parameters τ and υ. Again, which of the currencies should be used to minimise output loss depends on the (τ, υ). The results are summarized on graphs 6a and 6b.